then (6) is over-all asymptotically stable, i.e. any of its solutions approach some stable equilibrium as $t \to +\infty$.

The estimate (11) is a generalization of a similar estimate by Iudovich /3/: $2\sigma_1 - b_1 < 0$ which was obtained using the Liapunov function.

*Note 1.* When $r_1 \to 1$, the right-hand side of inequality (11) approaches $+\infty$. Hence for fixed $b_1$ and $\sigma_1$ an $r_1 > 1$, will always be found that satisfies condition (11).

*Note 2.* Sometimes it is interesting to consider small $b_1$ /4/. Selecting in that case $\lambda = b_1/2$ we obtain from (11) the following condition of global asymptotic stability:

$$r_1 < 1 + \frac{(\sigma_1 + 1)}{\sqrt{2\sigma_1}} b_1$$

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# THE LAWS OF VARIATION OF ENERGY AND MOMENTUM FOR ONE-DIMENSIONAL SYSTEMS WITH MOVING MOUNTINGS AND LOADS*

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The self-consistent dynamic behaviour of a one-dimensional system with a moving load is considered. The natural boundary conditions, previously obtained from the variational Hamiltonian principle /1/ for the self-consistent problem of the dynamics of one-dimensional systems, when the boundary motions are not specified, are used to show that the motion of the load results in the appearance of additional forces that are proportional to the load velocity. Expressions are obtained in terms of the density of the Lagrange function for the wave pressure, the wave energy flux, the wave momentum, the energy transport velocity, the work of the forces that vary the system parameters, and the distributed recoil forces that occur when waves propagate along a non-uniform system. The radiation conditions are discussed in the class of systems considered. The critical velocities of the load moving along a Timoshenko beam are determined.

1. Consider a holonomic system with ideal constraints, consisting of a one-dimensional system along which a concentrated load is displaced, consistent with the motion of a distributed system.

Let $x$ be a Cartesian coordinate along the one-dimensional system, $t$ be the time, and $D = \{(x, t) : a \leqslant x \leqslant b,\ \alpha \leqslant t \leqslant \beta\}$ be some rectangular region in the plane $x, t$. We assume that the motion of the load is defined by some function $x = \chi(t)$, doubly differentiable in $[\alpha, \beta]$, such that the curve $K = \{(x, t): x = \chi(t),\ \alpha \leqslant t \leqslant \beta\}$ lies in region $D$ and divides it into parts as follows:

$$D_1 = \{(x, t):\ a \leqslant x \leqslant \chi(t),\quad \alpha \leqslant t \leqslant \beta\},\qquad D_2 = \{(x, t): \chi(t) \leqslant x \leqslant b,\ \alpha \leqslant t \leqslant \beta\}$$

and that the law of motion of the distributed system is defined by some vector function continuous in $D$

$$\mathbf{u}(x, t) = \begin{cases} \mathbf{u}^1(x, t) = (u_1^1(x, t), \ldots, u_n^1(x, t)), & (x, t) \in D_1 \\ \mathbf{u}^2(x, t) = (u_1^2(x, t), \ldots, u_n^2(x, t)), & (x, t) \in D_2 \end{cases}$$

where the vector functions $\mathbf{u}^1(x, t)$ and $\mathbf{u}^2(x, t)$ are doubly continuously differentiable in regions $D_1$ and $D_2$, respectively.

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Thus curve $K$ is a possible line of discontinuity of the vector function $\mathbf{u}(x, t)$ in region $D$, and, conversely, the discontinuity line of the derivatives of the vector function $\mathbf{u}(x, t)$ in region $D$ can only be the line $K$.

Let $L(t, x, \mathbf{u}, \mathbf{u}_t, \mathbf{u}_x)$ be the density of the Lagrange functions of the distributed system, and $L^\circ(t, \chi, \chi^\cdot, \mathbf{u}^\circ, \mathbf{u}^{\cdot\circ})$ be the Lagrange function of the load ($L$ and $L^\circ$ are doubly differentiable functions $\mathbf{u}^\circ(t) = \mathbf{u}(\chi(t), t)$ of their arguments). Then the relations that specify the equations of motion of the system and conditions on the moving boundary have the form

$$L_u - \frac{\partial}{\partial t} L_v - \frac{\partial}{\partial x} L_w + \mathbf{q} = 0, \quad (x, t) \in \operatorname{Int} D_\nu, \quad \nu = 1, 2 \tag{1.1}$$

$$L_\chi^\circ - \frac{d}{dt} L_{\chi^\cdot}^\circ + q_0^\circ = [L - (\mathbf{w}, L_w - \chi^\cdot L_v)] \tag{1.2}$$

$$[\mathbf{u}] = 0 \tag{1.3}$$

$$L_{u^\circ}^\circ - \frac{d}{dt} L_{u^{\cdot\circ}}^\circ + \mathbf{q}^\circ = [L_w - \chi^\cdot L_v] \tag{1.4}$$

where for brevity the scalar product of the vectors is denoted by $\mathbf{v} = \mathbf{u}_t, \mathbf{w} = \mathbf{u}_x, (\cdot, \cdot)$, and $[\cdot]$ is the difference between the limit values of the functions on the two sides of curve $K$. Here, the dissipative effects caused by the dissipating forces $\mathbf{q}, \mathbf{q}^\circ, \mathbf{q}_0^\circ$ are taken into account.

Note that, as applied to a number of special cases, the question of correct conditions at the moving boundaries were considered in /2-4/.

Formulas (1.4) that express the balances of generalized forces acting on the load show that, unlike the stationary load, the moving load is subjected to inertia forces - components of the vector $\chi^\cdot L_v$ - proportional to the velocity of motion $\chi^\cdot$. In view of the fact that in the dynamic theory of one-dimensional elastic systems these forces were, as a rule, neglected /5-7/, it is interesting to estimate their magnitude. For instance, for pit elevator cables, where the velocity of transverse waves is 20-70 m/sec and the elevator velocity is $\chi^\cdot = 8-12$ m/sec, the ratio of the inertia forces to the shear forces is 0.1-0.4. It is obvious that, when calculating the dynamic behaviour of such systems, allowance must be made for these forces.

Expressions for the inertia forces can be obtained not only from variational principles. This can be shown using the example of an absolutely smooth string on which a bead of mass $m$ slides at constant velocity. To determine the shear force acting on the moving bead we will write the equation of the string transverse oscillations in a coordinate system attached to the bead

$$\frac{\partial}{\partial t^*} (\rho (u_{t*} - c u_{x*})) = \frac{\partial R}{\partial x^*}, \quad R = (N - c^2\rho) u_{x*} + c\rho u_{t*}$$

where $x^*$ and $t^*$ are connected with the original coordinates by the Galilean transformation $x^* = x - ct,\ t^* = t$. On the left-hand side we have here the rate of variation of the momentum density, and on the right-hand side the gradient of the shear force $R$. Knowing it, we can construct the equation of transverse motion of the bead. Reverting to the original coordinates, we obtain

$$mu^{\cdot\cdot}(ct, t) = (Nu_x + c\rho u_t)|_{x=ct+0} - (Nu_x + c\rho u_t)|_{x=ct-0}$$

where $c\rho u_t$ is the force of inertia.

It is interesting to note that in electrodynamics, when solving problems with moving boundaries, formulas for the transformation of the electric field $E$ and the magnetic field $H$ are used in accordance with the special theory of relativity /8/ and, consequently, the boundary conditions are always obtained in the form (1.4). Indeed, by introducing the single-scalar definition for plane electromagnetic waves, setting $H = -u_x/\mu,\ E = u_t$, and taking into account that $L = (\varepsilon E^2 - \mu H^2)/2$ ($\varepsilon$ and $\mu$ are the permittivity and permeability respectively), from Eq.(1.4) we obtain, for example, the condition for the fields on a moving discontinuity of the parameters ($\varepsilon$ and $\mu$) of the medium $[H - \chi^\cdot \varepsilon E] = 0$, given in /9,10/.

When there are no external and dissipative forces ($q_0^\circ = 0$) the motion of the load occurs due to the action of wave pressure forces on each side of the curve $K$, which by Eq.(1.2), are expressed by the formulas

$$L - (\mathbf{w}, L_w) + \chi^\cdot (\mathbf{w}, L_v)|_{x=\chi(t)\pm 0}$$

The magnitude of these forces may be considerable even in the case of a fixed boundary. For instance, for a circular bar executing flexural oscillations, a stress $\sigma \approx 10^3 (d^2/\lambda^4) E u_0^2$ occurs in the mountings ($d$ is the bar diameter, $\lambda$ is the wavelength, $E$ is the modulus of elasticity and $u_0$ the amplitude of transverse deflections). In particular, when $u_0 = d$ and $\lambda = 10d$ the stress is $\sigma \approx 10^{-1} E$, i.e. it is comparable with the ultimate stress admissible for metals.

If the load moves, the wave pressure force may increase. Thus, for a string its magnitude can be considerable, even if the intensity of the waves impinging on the wave boundary is as small as desired /11/.

2. Using (1.1) we can show that in each region $\mathrm{Int}\, D_\nu$ $(\nu = 1, 2)$ the following relations hold:

$$\frac{\partial h}{\partial t} + \frac{\partial S}{\partial x} = -L_t + (\mathbf{v}, \mathbf{q}), \quad h = (\mathbf{v}, L_v) - L, \quad S = (\mathbf{v}, L_w) \tag{2.1}$$
$$\frac{\partial p}{\partial t} + \frac{\partial T}{\partial x} = L_x - (\mathbf{w}, \mathbf{q}), \quad p = -(\mathbf{w}, L_v), \quad T = L - (\mathbf{w}, L_w)$$

where $h(x, t)$ is the density of the Hamiltonian function of the distributed system, and $T(x, t)$ is the wave pressure force at cross section $x$, as follows from (1.2).

For a stationary system $(L_t = 0)$ when there are no external and dissipative forces $(\mathbf{q} = 0)$ the first of formulas (2.1) represents the energy variation in an element of the distributed system due to its flow through the boundary of the element. Consequently, the quantity $S(x, t)$ must be regarded as the wave-energy flux. Similarly the quantity $p(x, t)$ is the wave momentum density. Thus Eqs.(2.1) are the local laws of variation of the wave energy and momentum.

Note that $V = S/h$ is the velocity of wave-energy transport (the group velocity /12/). It can be shown that, for instance, in the case of a stretched string, for quasiharmonic waves the mean volocity $V$ over a period is $d\omega/dk$ ($\omega$ is the frequency and $k$ is the wave number), which is the same as the approximate expression for the group velocity /13/.

The functions $h(x, t), S(x, t), p(x, t), T(x, t)$ have a discontinuity in the region $D$ along the curve $K$. Using (1.2) and (1.4) it can be shown that the respective conditions at the discontinuity, which are essentially the laws of variation of the load energy and momentum, have the form

$$\begin{aligned} dh^\circ/dt &= -[S - \chi^{\cdot} h] - L_t^\circ + (\mathbf{u}^{\cdot\circ}, \mathbf{q}^\circ) + \chi^{\cdot} q_0^\circ \\ dp^\circ/dt &= -[T - \chi^{\cdot} p] + L_\chi^\circ + q_0^\circ \\ h^\circ &= \chi^{\cdot} L_{\chi^{\cdot}}^\circ + (\mathbf{u}^{\cdot\circ}, L_{u^{\cdot\circ}}) - L^\circ, \quad p^\circ = L_{\chi^{\cdot}}^\circ \end{aligned} \tag{2.2}$$

where $h^\circ$ and $p^\circ$ are the Hamiltonian function and momentum of the load respectively.

The following global laws of variation of the energy and momentum for the system as a whole hold:

$$\frac{dH}{dt} = -S\,|_{x=a}^{x=b} - \langle L_t \rangle - L_t^\circ + \langle (\mathbf{v}, \mathbf{q}) \rangle + (\mathbf{u}^{\cdot\circ}, \mathbf{q}^\circ) + \chi^{\cdot} q_0^\circ \tag{2.3}$$

$$\frac{dP}{dt} = -T\,|_{x=a}^{x=b} + \langle L_x \rangle + L_\chi^\circ - \langle (\mathbf{w}, \mathbf{q}) \rangle + q_0^\circ \tag{2.4}$$

where $H = \langle h \rangle + h^\circ$ and $P = \langle p \rangle + p^\circ$ are the Hamiltonian function and momentum of the complete system, respectively, and the angular brackets indicate integration with respect to $x$ from $a$ to $b$.

It is seen from Eqs.(2.3) that an increase in wave energy in the case of fixed boundaries is only possible when the forces do positive work that changes the system parameters, i.e. when $-(\langle L_t \rangle + L_t^\circ) > 0$. A change in momentum, as follows from (2.4), can occur not only because of the pressure $T$ at the boundaries $x = a$ and $x = b$ of the elastic force $L_\chi^\circ$, and because of the dissipative and external forces, but also due to the recoil forces $\langle L_x \rangle$, resulting from distributed reflection of the waves as they propagate along an inhomogeneous system.

To investigate the radiation conditions the first condition at the discontinuity (2.2) can be used. The expressions $(S - \chi^{\cdot} h)\,|_{x=\chi(t)\pm 0}$ appearing here define the wave-energy flux on each side of curve $K$. Hence the criterion for radiation to occur into a distributed system with a moving load can be written in the form

$$|\,[S - \chi^{\cdot} h]\,| > 0 \tag{2.5}$$

*Example 1.* Consider an infinitely long line through which, at cross section $x = \chi(t)$ a given current $I$ flows. The density of the Lagrange function for the line is written in the form $L = (CV^2 - \Lambda J^2)/2$, where $V$ and $J$ are the voltage and current in the line, and $C$ and $\Lambda$ are the capacitance and inductance per unit length. Changing to the single-scalar definition, setting $J = -\Lambda^{-1} u_x$, $V = u_t$ and using (2.5), we find that radiation occurs, when

$$|\,[S - \chi^{\cdot} h]\,| = |\,\Lambda^{-1} \chi^{\cdot -1} (1 - C\Lambda \chi^{\cdot 2})\,[u_t^2]/2\,| > 0$$

Hence follows the well-known condition for the source velocity /14,15/ for which Cherenkov radiation occurs: $\chi^{\cdot} = \pm (C\Lambda)^{-1/2}$, where $(C\Lambda)^{-1/2}$ is the velocity of propagation of electromagnetic waves in the line.

*Example 2.* For the Timoshenko beam we have

$$L = (\rho F u_t^2 + \rho J \varphi_t^2 - EJ \varphi_x^2 - \varkappa GF (u_x - \varphi)^2 - N u_x^2)/2$$

Here $u(x, t)$ is the transverse deflection of the middle line of the beam, $\varphi(x, t)$ is the angle of rotation of the cross section, $\rho$ is the specific density, $F$ is the area of cross section, $J$ is the moment of inertia of the section, $E$ and $F$ are the moduli of elasticity, $\varkappa$ is the Timoshenko coefficient, and $N$ is the strain.

By Eq.(2.5) one of the beam critical velocities (which correspond to the boundaries of the radiation region), is

$$\chi_1^{\cdot} = \pm ((N + \varkappa G\rho)/(\rho F))^{1/2}$$

If, in addition, it is assumed that the load moves at uniform velocity and the solutions on the left and right of it are harmonic, two more critical velocities are defined

$$\chi_2^{\cdot} = \pm (E/\rho)^{1/2}, \ \chi_3^{\cdot} = \pm ((N/(\rho F)) + (2\varkappa G/(3\rho)))^{1/2}$$

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# ON THE LAW OF ANGULAR MOMENTUM VARIATION OF A SPHERE ROLLING ON A STATIONARY SURFACE*

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The rolling of a homogeneous sphere without friction on a stationary surface is considered. The forms of surfaces are established, and the axes corresponding to them, relative to which the sphere angular momentum variation is determined, are defined by the same differential equation, as if the axes were stationary.

1. The theorem of the variation of the angular momentum $\mathbf{K}_A$ of a mechanical system relative to an arbitrary pole $A$ has the form /1/

$$\mathbf{K}_A^{\cdot} + M\mathbf{v}_A \times \mathbf{v}_G = \sum \mathrm{mom}_A \mathbf{R} + \sum \mathrm{mom}_A \mathbf{F} \qquad (1.1)$$

Here $\mathbf{v}_A$ is the velocity of the point $A$ in fixed space, $M\mathbf{v}_G$ is the momentum of the body, and the right-hand side of (1.1), is the sum of the principal moments about the point $A$ of the constraint reactions and the active forces operating on the system.

Suppose some axis $AL$ constantly pass through the moving point $A$, and let $\mathbf{e}$ be the unit vector of that axis. If the constraints at each instant of time allow a virtual rotation of the system as a single rigid body about the $AL$ axis and the kinematic condition

$$M(\mathbf{v}_G, \mathbf{v}_A \times \mathbf{e}) + (\mathbf{K}_A, \mathbf{e}^{\cdot}) = 0 \qquad (1.2)$$

is satisfied /2/, then from (1.1) we have the scalar equation

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